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THE SOLUTION OF A CLASS OF TWO-DIMENSIONAL MELTING AND SOLIDIFICATION PROBLEMS

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Abstract—Two-dimensional melting and solidification problems are divided into three convenient classes and a general method of solution for one of these is presented. The method utilizes the concept of a fictitious body of constant geometry, in which is embedded the real body (whose dimensions are changing due to the change in phase); the fictitious body is acted upon by a fictitious heat flux or (in some problems) a fictitious initial temperature distribution. The problem thus formulated in terms of these unknown fictitious quantities results in a set of integro-differential equations to be solved simultaneously, either numerically or in series form. Two problems are considered in detail. In the first problem the melting of a finite, insulated slab with the melt immediately removed is formulated and an example for a semi-infinite, insulated strip is given. In the second problem the solidification of a finite, insulated slab with zero superheat is formulated and an example for a specific cooling history is given. In both problems two-dimensional effects are introduced by spatial variations of heating or cooling conditions, and short-time series solutions are developed.

1. INTRODUCTION

PROBLEMS of heat conduction in which a body undergoes a phase change during exposure to a thermal environment have received a great deal of attention in recent years. Almost all the problems which have been solved, however, have been restricted to one-dimensional cases. The present paper deals with two-dimensional melting and solidification problems: it presents a convenient classification of such problems, and a general method of solution for a certain class. Two examples of problems in this class are treated; the first is an ablation problem (the melt being immediately removed upon formation), the second is a problem motivated by the solidification of castings. Illustrative problems for the other classes are discussed.

The first published work on change-of-phase problems is that of Stefan [1] in his study of the thickness of polar ice. A more general treatment, however, including the most important known exact solution, that for the half-space $z > 0$ under prescribed boundary temperature was given earlier (in the 1860's) by Neumann. Neumann's solution is characterized by similarity, that is, it is a function of the single variable $(z/\sqrt{t})$, the motion of the solid–liquid interface being proportional to $\sqrt{t}$. Several other solutions of this type have been found [2, 3] but for boundary conditions such as those of prescribed flux or radiation, solutions of this type are not possible.

Because of the difficulties of the problem, numerical solutions (for example [4–7]) and approximate analytical techniques (for example [8–10]) have been extensively used in the literature.† The above cited works refer to one-dimensional problems, with the

† For an extensive bibliography of papers in all aspects of heat conduction see [3].

exception of that of references [7] and [10]. In [7], Springer and Olson have developed a finite difference scheme for the case of a multi-phase, axisymmetric, finite tube, including variable thermal properties and various boundary conditions. Poots [10] treats the two-dimensional problem of the solidification of a liquid square initially at the melting temperature T_M and with all boundaries at a fixed temperature $T < T_M$. Assumptions are made as to the general shape of the interface and as to the temperature distribution, and the solution is obtained by means of the heat balance integral.

A different approach, which can be used both for numerical analyses and for the construction of short-time solutions† in series form, has been recently introduced [11, 3]. In this approach one deals mathematically with a fictitious body of known constant geometry instead of the actual one whose boundary is unknown. An unknown fictitious heat flux, and sometimes a fictitious initial temperature distribution are introduced. The temperature field is obtained in terms of the unknown heat flux, and is then used to satisfy the necessary interface conditions at some unknown location (corresponding to the actual moving boundary) in the interior of the body. The original partial differential boundary value problem is thus replaced by a coupled integro-differential problem expressed in terms of the unknown heat flux and interface position.

The present paper presents an extension of this approach to two-dimensional problems. Section 2 gives a general discussion of such problems, and presents a convenient classification in which three classes of two-dimensional problems are identified. For two of these classes, illustrative solutions are constructed by means of an inverse process. Complete solutions for an ablation and a solidification problem of the remaining class appear in Sections 3 and 4, respectively.

2. GENERAL DISCUSSION AND CLASSIFICATION OF PROBLEMS

2.1. *Interface conditions*

The principal conditions which characterize change-of-phase problems are those which hold at the interface separating the liquid and the solid regions. Let the equation of this surface, in a two-dimensional problem, be $z = \mathscr{A}(x, t)$. Separate heat-conduction problems exist in the liquid and the solid regions which are of course unknown since the moving interface is unknown *a priori* and is one of the quantities to be determined. The two problems, however, are coupled through the two interface conditions. The temperature in both liquid and solid regions (T_L, T_S, respectively) must be equal to the melting temperature T_m there, and the difference in heat flux between the liquid and solid region must be equal to the rate of heat absorbed per unit area during the phase change; this rate, in turn, is proportional to the velocity of the interface. Mathematically these conditions are

$$T_L = T_S = T_m, \tag{1}$$

$$\text{at } z = \mathscr{A}(x, t)$$

$$\left[1+\left(\frac{\partial \mathscr{A}}{\partial x}\right)^2\right]\left[k_S\frac{\partial T_S}{\partial z}-k_L\frac{\partial T_L}{\partial z}\right] = \pm\rho_S L\frac{\partial \mathscr{A}}{\partial t}, \tag{2}$$

where k_L, k_S are thermal conductivities of liquid and solid, ρ_S is the density of the solid, and L is the latent heat of fusion. Equation (2) holds for either melting or solidification;

† Short-time solutions, sometimes referred to as starting solutions, are not only of interest in themselves but are often useful for the start of numerical or approximate analyses; cf. Citron [12].

in either case, the positive sign is to be used if motion of the interface into the solid corresponds to a positive value of $(\partial\phi/\partial t)$, and the negative sign if motion of the interface towards the liquid corresponds to a positive $(\partial\phi/\partial t)$. In the derivation of equation (2) the heat-balance condition was simplified by the use of equation (1); see [7, 21].

2.2. *Criterion for the onset of melting*

It is desired to devise a criterion for determining when and where a phase change occurs on the surface of the body, only problems in which the phase change originates at a boundary and proceeds into the region being considered. Clearly, no melting occurs as long as $T < T_m$ throughout; but if a time is found after which a solution obtained neglecting the change of phase somewhere exceeds T_m, then melting will have started and this solution is no longer valid. Let the solution obtained neglecting melting be denoted by $T_N(P, t)$. Melting then starts at $t = t_m$ if and only if†

$$T_N(P, t) \leq T_m \qquad \text{for all } P, t \leq t_m, \tag{3a}$$

$$T_N(P_B, t_m) = T_m \qquad \text{for some } P_B \text{ on the boundary,} \tag{3b}$$

and a number $\varepsilon > 0$ exists such that

$$T_N(P_B, t) > T_m \qquad \text{for } t_m < t \leq t_m(1+\varepsilon). \tag{3c}$$

If the temperature on the boundary is continuous, then equation (3c) is obviously satisfied whenever

$$\frac{\partial T_N}{\partial t}(P_B, t_m) > 0. \tag{3d}$$

Such will be the case in all problems considered in this paper.

If the prescribed surface conditions contain only continuous inhomogeneous terms, then the use of criterion (3) is straightforward, since the solution obtained before melting is indeed T_N. If, however, a discontinuity in these terms occurs (e.g. the example given below, or that of Section 3, in which a discontinuous heat input is applied), then the application of equation (3d) is rather laborious in two-dimensional problems. However, it can be greatly simplified by noting that the initial effect of an arbitrary boundary flux is one-dimensional, i.e. $(\partial T_N/\partial t)(P_B, t)$ is initially the same whether the actual heat flux is used or is replaced by a flux constant in space and equal to the actual flux at that point, see [21].

As a specific example of the use of this criterion, consider the half space $z > 0$, initially at zero temperature, with the following heating history:

$$-k\frac{\partial T_N}{\partial z}(0, t) = \begin{cases} Q_0 & 0 < t < t' \\ Q_0 + Q_1 & t \geq t' \end{cases} \tag{4a}$$

where Q_0 and Q_1 are constant heat inputs.

† In solidification problems all inequalities in equations (3) are of course reversed.

The solution is

$$T_N(z,t) = \begin{cases} \dfrac{2Q_0\sqrt{(\kappa t)}}{k}\,\mathrm{ierfc}\dfrac{z}{2\sqrt{(\kappa t)}}; & 0 < t < t' \\ \dfrac{2Q_0\sqrt{(\kappa t)}}{k}\,\mathrm{ierfc}\dfrac{z}{2\sqrt{(\kappa t)}} + \dfrac{2Q_1\sqrt{[\kappa(t-t')]}}{k}\,\mathrm{ierfc}\dfrac{z}{2\sqrt{[\kappa(t-t')]}}; & t \geq t' \end{cases} \tag{4b}$$

where $\kappa = k/\rho_S c$ and c is the specific heat.
Let t' be the time at which $T_N(0, t') = T_m$, that is,

$$t' = \frac{\pi}{\kappa}\left(\frac{kT_m}{2Q_0}\right)^2. \tag{4c}$$

It is desired to determine whether or not $t' = t_m$. Equations (3a) and (3b) are satisfied at $t = t'$, while

$$\frac{\partial T_N}{\partial t}(0,t) = \frac{Q_0}{k}\sqrt{\left(\frac{\kappa}{\pi t}\right)}+\frac{Q_1}{k}\sqrt{\left[\frac{\kappa}{\pi(t-t')}\right]}; \qquad t > t'. \tag{5a}$$

Comparison with (3d) shows that $t' = t_m$ if $Q_1 \geq 0$, while $t' \neq t_m$ if $Q_1 < 0$. Indeed, in the latter case, with $\alpha = \sqrt{(t_m/t')}$,

$$\frac{Q_1}{Q_0} = -\sqrt{\left(\frac{\alpha-1}{\alpha+1}\right)} \qquad \text{or} \qquad t_m = t'\left(\frac{Q_0^2+Q_1^2}{Q_0^2-Q_1^2}\right)^2, \tag{5b}$$

with $T = T_m$ at $t = t'$, t_m and $T < T_m$ for $t' < t < t_m$.

2.3. *Initial rate of melting*

Consider the half space $z > 0$, with a heat flux $Q(t)$ on the boundary $z = 0$, and the melt instantaneously removed upon formation. A heat flux balance just before and just after melting results in the equations:

$$\begin{gathered} -k\frac{\partial T}{\partial z}(0, t_m^-) = Q_-(t_m^-), \\ -k\left[1+\left(\frac{\partial \mathscr{s}}{\partial x}\right)^2\right]\frac{\partial T}{\partial z}(\mathscr{s}, t_m^+) = Q_+(t_m^+) - \rho_S L\frac{\partial \mathscr{s}}{\partial t}(t_m^+). \end{gathered} \tag{6}$$

Letting $t_m^- \to t_m$, $t_m^+ \to t_m$ and subtracting the two equations, one obtains

$$\rho_S L\frac{\partial \mathscr{s}}{\partial t}(t_m) = Q_+(t_m) - Q_-(t_m) = \mathscr{S}(Q) \tag{7a}$$

where $\mathscr{S}(Q)$ is the jump in flux at $t = t_m$. Clearly, if $\mathscr{S}(Q) < 0$ melting will not start. Hence

$$\begin{aligned} &\text{if } \mathscr{S}(Q) < 0 && \text{no melting;} \\ &\text{if } \mathscr{S}(Q) = 0 && \frac{\partial \mathscr{s}}{\partial t}(t_m) = 0; \\ &\text{if } \mathscr{S}(Q) > 0 && \frac{\partial \mathscr{s}}{\partial t}(t_m) = \frac{\mathscr{S}(Q)}{\rho_S L} \end{aligned} \tag{7b}$$

or

$$\mathscr{S}(t) = \frac{\mathscr{S}(Q)}{\rho_S L}(t-t_m)+\ \dots\ ; \qquad t-t_m \ll t_m.$$

As an example consider the problem of Section 2.2; the short time solution is, from [11],

$$\mathscr{S}(t) = \frac{4Q_0}{3\pi\rho_S L\sqrt{t_m}}(t-t_m)^{3/2}+\dots; \qquad t-t_m \ll t_m, \tag{8a}$$

for the case in which $Q_1 = 0$, and, from equation (40),

$$\mathscr{S}(t) = \frac{Q_1}{\rho_S L}(t-t_m)+\ \dots; \qquad t-t_m \ll t_m, \tag{8b}$$

for the case in which $Q_1 > 0$.

In view of the fact that the initial behavior is essentially one-dimensional, equation (7b) holds for two-dimensional problems as well.

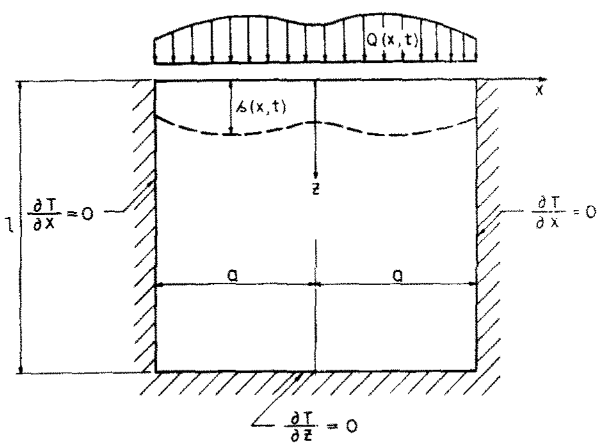

FIG. 1. The finite melting or solidifying slab.

2.4. *Classification of two-dimensional melting problems*

It is convenient to classify two-dimensional melting problems according to the initial behavior of the melt region.† Considering the slab of Fig. 1, the following three categories may be distinguished in order of apparent increasing analytical difficulty.

(1) Melting begins simultaneously at $t = t_m$ for all $|x| \le a$. All one-dimensional problems belong to this classification.

(2) Melting begins in a number of segments of non-zero length Δx_i in the region $|x| \le a$, with

$$\sum_1^n \Delta x_i < 2a. \qquad i = 1, 2, \dots, n.$$

(3) Melting begins at one isolated point at least in the region $|x| \le a$.

† Note that melting starts at the surface in problems such as those considered here, in which no internal heat generation is present.

Simple illustrations of problems in these categories are the following: Consider a semi-infinite strip initially solid at the melting temperature T_m. If a heat flux positive for all $|x| \leq a$ is imposed on the boundary at a time $t = 0$, melting begins immediately for all $|x| \leq a$ and the problem is of class (1). If the heat flux is part positive and part negative melting initially occurs only for those portions of the boundary where the flux is positive; the problem is then of class (2). A point source of heat applied to the boundary results in initial melting at that point only, and hence this problem belongs to the third category.

The present investigation is concerned with two-dimensional problems of class (1), for which a general method of solution is developed in Sections 3 and 4. Illustrative solutions to problems of classes (2) and (3) will, however, be constructed in a straightforward manner by an inverse process in Sections 2.5 and 2.6,† as follows: The concept mentioned in the introduction, of a melting body extended mathematically to its original dimensions is used. One assumes the fictitious heat flux on the surface of the extended body to be known. The temperature condition (1) results in a transcendental equation for the melting thickness $\mathscr{s}(x, t)$, and from the heat balance condition at $\mathscr{s}(x, t)$ one obtains the real heat flux.

2.5. *Example of a problem of class (2)*

Consider a semi-infinite, insulated strip (geometry shown in inset of Fig. 2), initially at zero temperature. Let the boundary $z = 0$ be heated as follows:

$$-k\frac{\partial T}{\partial z}(x, 0, t) = \begin{cases} Q_0 & 0 < t < t_m \\ Q_0\left(1+\cos\dfrac{\pi x}{a}\right) & t \geq t_m \end{cases} \tag{9}$$

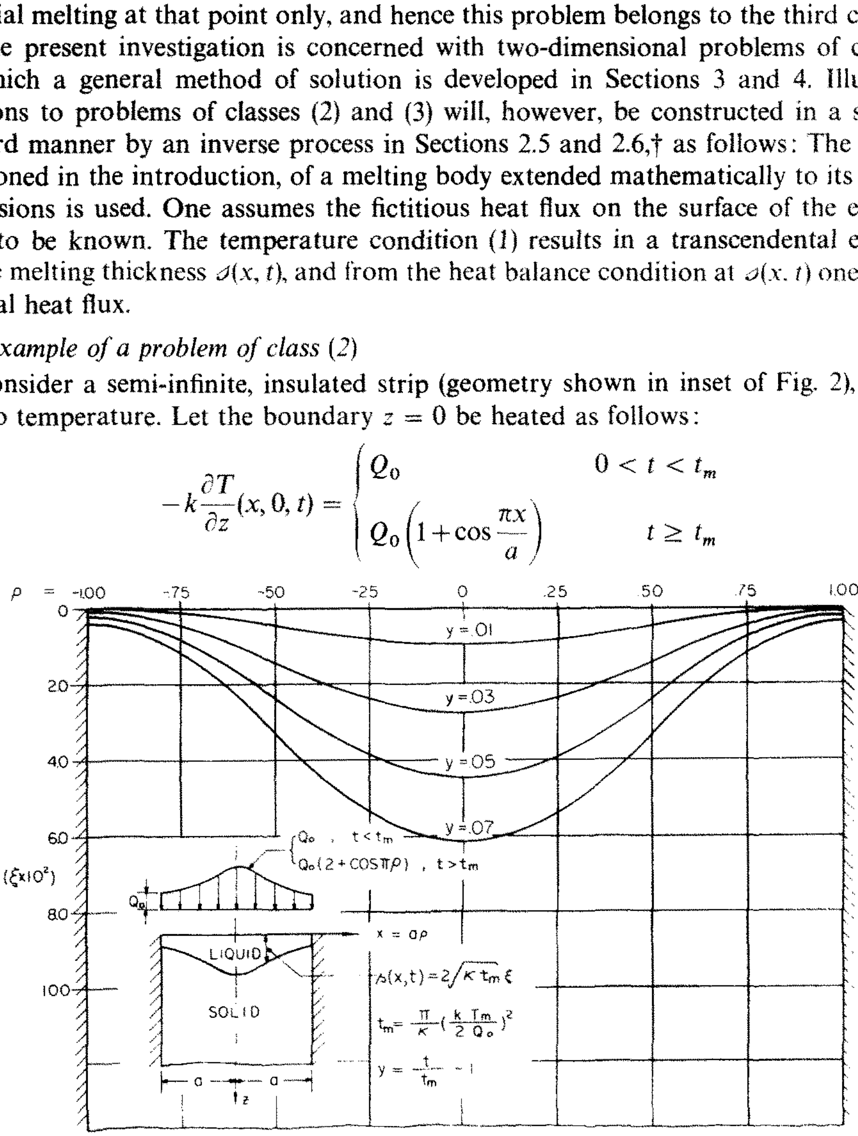

FIG. 2. A dimensionless plot of the melt thickness vs. x/a for various values of time and for $m = n = C_0 = C_1 = 1$.

† Such inverse solutions are of more than academic interest, since they often prove useful in the construction of upper and lower bounds to the solution of a more complex problem for which no analytical solution is known. Such bounds have been used in [2, 13–15] for one-dimensional problems, and in [16] for two- and three-dimensional problems.

where for $t \geq t_m$, $Q_0[1+\cos(\pi x/a)]$ represents the real flux where $\delta = 0$ and the fictitious flux where $\delta \neq 0$.

The temperature field is [17]:

$$T(x, z, t) = \frac{2Q_0\sqrt{(\kappa t)}}{k} \operatorname{ierfc} \frac{z}{2\sqrt{(\kappa t)}} + \frac{Q_0 a}{2\pi k}\left\{e^{-(\pi z/a)}\left[1+\operatorname{erf}\left(\frac{\pi}{a}\sqrt{[\kappa(t-t_m)]}-\frac{z}{2\sqrt{[\kappa(t-t_m)]}}\right)\right]\right. \quad (10a)$$

$$\left. - e^{(\pi z/a)}\left[1-\operatorname{erf}\left(\frac{\pi}{a}\sqrt{[\kappa(t-t_m)]}+\frac{z}{2\sqrt{[\kappa(t-t_m)]}}\right)\right]\right\}\cos\frac{\pi x}{a},\ t \geq t_m,$$

where

$$T(x, 0, t_m) = T_m = \frac{2Q_0\sqrt{(\kappa t_m)}}{\sqrt{(\pi)}k}. \quad (10b)$$

It is convenient to use the following non-dimensional variables and parameters:

$$y = \frac{t-t_m}{t_m}; \qquad \rho = \frac{x}{a}; \qquad \zeta = \frac{z}{a};$$

$$\xi = \frac{\delta}{2\sqrt{(\kappa t_m)}}; \qquad n = \frac{a}{2\sqrt{(\kappa t_m)}}; \qquad m = \frac{Q_0\sqrt{(t_m)}}{\rho_S\sqrt{(\kappa)}L}; \quad (11)$$

so that the temperature field can be written as

$$\frac{T(\rho, \zeta, y)}{T_m} = \sqrt{[\pi(y+1)]} \operatorname{ierfc}\frac{n\zeta}{\sqrt{(y+1)}} + \frac{n}{2\sqrt{\pi}}\left\{e^{-\pi\zeta}\left[1+\operatorname{erf}\left(\frac{\pi}{2n}\sqrt{y}-\frac{n\zeta}{\sqrt{y}}\right)\right]\right. \quad (12)$$

$$\left. - e^{\pi\zeta}\left[1-\operatorname{erf}\left(\frac{\pi}{2n}\sqrt{y}+\frac{n\zeta}{\sqrt{y}}\right)\right]\right\}\cos\pi\rho; \quad y \geq 0.$$

Along the melt line $z = \delta$ or $n\zeta = \xi$ the temperature must be the melting temperature T_m. Therefore

$$1 = \sqrt{[\pi(y+1)]} \operatorname{ierfc}\frac{\xi}{\sqrt{(y+1)}} + \frac{n}{2\sqrt{\pi}}\left\{e^{-(\pi\xi/n)}\left[1+\operatorname{erf}\left(\frac{\pi}{2n}\sqrt{y}-\frac{\xi}{\sqrt{y}}\right)\right]\right. \quad (13a)$$

$$\left. - e^{(\pi\xi/n)}\left[1-\operatorname{erf}\left(\frac{\pi}{2n}\sqrt{y}+\frac{\xi}{\sqrt{y}}\right)\right]\right\}\cos\pi\rho, \qquad y \geq 0, \quad \xi \geq 0.$$

This is a transcendental equation for the non-dimensional melting thickness $\xi(\rho, y)$. For short times it can be shown,† see [21], that a solution of the form

$$\xi = A(\rho)\sqrt{y} + \ldots, \qquad y \ll 1 \quad (13b)$$

satisfies equation (13a) where $A(\rho)$ is the solution to the transcendental equation

$$\frac{A(\rho)}{\operatorname{ierfc} A(\rho)} = \cos\pi\rho, \qquad A(\rho) \geq 0. \quad (13c)$$

The initial extent of the melted region in the x-direction is obtained by setting $A = 0$, which yields $\rho = \frac{1}{2}$. Hence the portion $|\rho| < \frac{1}{2}$ melts initially, and the portion $\frac{1}{2} \leq |\rho| \leq 1$ does not. To calculate the growth of the melted portion with time, consider equation

† By verifying that, if $\xi = A(\rho)y^n$, $n \neq \frac{1}{2}$ leads to a contradiction.

(13a) written at the extreme point of this region, i.e. where $\xi = 0$ and where $\rho = \rho_0(y)$, say. The result is

$$1 = \sqrt{(y+1)}+\frac{n}{\sqrt{\pi}}\operatorname{erf}\frac{\pi}{2n}\sqrt{y}\cos[\pi\rho_0(y)], \tag{14a}$$

which for small times yields

$$\rho_0(y) = \frac{1}{\pi}\cos^{-1}\left(-\frac{\sqrt{y}}{2}\right) = \frac{1}{2}+\frac{\sqrt{y}}{2\pi}+\ldots, \qquad y \ll 1. \tag{14b}$$

Therefore, the melted region initially grows in the ρ-direction as $\sqrt{(y)}/2\pi$, and initially, as before, $\rho_0(0) = \frac{1}{2}$. Therefore, this is indeed a problem of class (2).

The real heat flux in the melt region can now be found from the heat balance condition, that is, with instantaneous removal of the liquid,

$$Q(x, t) = \rho_S L\frac{\partial s}{\partial t}-k\left[1+\left(\frac{\partial s}{\partial x}\right)^2\right]\frac{\partial T}{\partial z} \qquad \text{at } z = s(x, t) \tag{15a}$$

which, with the temperature (10a) and non-dimensional form, becomes

$$\begin{aligned}\frac{Q}{Q_0}(\rho, y) = \frac{2}{m}\frac{\partial\xi}{\partial y}+\left\{1+\frac{1}{n^2}\left(\frac{\partial\xi}{\partial\rho}\right)^2\right\}\Bigg\{&\operatorname{erfc}\frac{\xi}{\sqrt{(y+1)}}\\ &+\frac{1}{2}\left[e^{-(\pi\xi/n)}\left(1+\operatorname{erf}\left[\frac{\pi}{2n}\sqrt{y}-\frac{\xi}{\sqrt{y}}\right]\right)+e^{\pi\xi/n}\left(1-\operatorname{erf}\left[\frac{\pi}{2n}\sqrt{y}+\frac{\xi}{\sqrt{y}}\right]\right)\right]\cos\pi\rho\Bigg\}.\end{aligned} \tag{15b}$$

Substituting from equation (13b) for the melt thickness and expanding for short times, one obtains the real heat flux as:

$$\frac{Q(\rho, y)}{Q_0} = \frac{A(\rho)}{m}\frac{1}{\sqrt{y}}+\ldots \qquad y \ll 1, \tag{15c}$$

where, again, $A(\rho)$ is the solution to (13c).

The real physical problem solved above is therefore the following. A constant heat flux Q_0 is imposed up to the time of melting ($t < t_m$). At $t = t_m$ the real heat flux jumps to infinity in the region $|\rho| \leq \frac{1}{2}$, while outside this region, i.e. for $\frac{1}{2} < |\rho| \leq 1$, the real heat flux is given by $Q_0(1+\cos\pi\rho)$. For later times ($t > t_m$) the real heat input decreases and melting continues over the region $|\rho| < \rho_0(y)$; in the region $\rho_0 < \rho \leq 1$, the heat input is still $Q_0(1+\cos\pi\rho)$. Note that the initially infinite melting rate corresponding to equation (13b) is in agreement with relations (7b).

2.6. *Example of a problem of class (3)*

Consider the semi-infinite strip of the preceding section, initially at zero temperature, heated as follows

$$-k\frac{\partial T}{\partial z}(x, 0, t) = Q_0\cos\frac{\pi x}{a}; \qquad t > 0, \tag{16}$$

both before and after melting. The temperature field is

$$T(x, z, t) = \frac{Q_0 a}{2\pi k}\left\{e^{-(\pi z/a)}\left[1+\operatorname{erf}\left(\frac{\pi}{a}\sqrt{(\kappa t)}-\frac{z}{2\sqrt{(\kappa t)}}\right)\right] - e^{\pi z/a}\left[1-\operatorname{erf}\left(\frac{\pi}{a}\sqrt{(\kappa t)}+\frac{z}{2\sqrt{(\kappa t)}}\right)\right]\right\}\cos\frac{\pi x}{a}. \tag{17a}$$

Since the temperature has its maximum value at $x = z = 0$, melting will begin there; thus $\rho_0(0) = 0$. The relation between T_m, t_m in this case is

$$T(0, 0, t_m) = T_m = \frac{Q_0 a}{\pi k}\operatorname{erf}\frac{\pi}{a}\sqrt{(\kappa t_m)}. \tag{17b}$$

With the non-dimensional variables and parameters of equations (11) the temperature field becomes

$$\frac{T}{T_m}(\rho, \zeta, y) = \frac{1}{2\operatorname{erf}(\pi/2n)}\left\{e^{-\pi\zeta}\left[1+\operatorname{erf}\left(\frac{\pi}{2n}\sqrt{(y+1)}-\frac{n\zeta}{\sqrt{(y+1)}}\right)\right] - e^{\pi\zeta}\left[1-\operatorname{erf}\left(\frac{\pi}{2n}\sqrt{(y+1)}+\frac{n\zeta}{\sqrt{(y+1)}}\right)\right]\right\}\cos\pi\rho. \tag{17c}$$

The melt thickness $\xi(\rho, y)$ can now be found as before by setting $T/T_m = 1$ on the melt line $z = \mathscr{A}$; thus

$$1 = \frac{1}{2\operatorname{erf}(\pi/2n)}\left\{e^{-(\pi\xi/n)}\left[1+\operatorname{erf}\left(\frac{\pi}{2n}\sqrt{(y+1)}-\frac{\xi}{\sqrt{(y+1)}}\right)\right] - e^{\pi\xi/n}\left[1-\operatorname{erf}\left(\frac{\pi}{2n}\sqrt{(y+1)}+\frac{\xi}{\sqrt{(y+1)}}\right)\right]\right\}\cos\pi\rho. \tag{18a}$$

Expanding this transcendental equation in $\xi(\rho, y)$ for short times and retaining first order terms in ξ, ρ and y only, one obtains

$$\xi(\rho, y) = \frac{1}{2\sqrt{\pi}}e^{-(\pi/2n)^2}y - \frac{\pi n}{2}\rho^2\operatorname{erf}\left(\frac{\pi}{2n}\right)+\ldots, |\rho| \leq \rho_0(y); \qquad \rho_0(0) = 0; \quad y \ll 1. \tag{18b}$$

The growth of the melted region $\rho_0(y)$ can be found by writing equation (18a) for $\xi = 0$ and $\rho = \rho_0(y)$, namely

$$\operatorname{erf}\frac{\pi}{2n} = \operatorname{erf}\left(\frac{\pi}{2n}\sqrt{(y+1)}\right)\cos\pi\rho_0(y). \tag{19a}$$

Again expanding for small y and retaining first order terms only, the width of the melted region is given by†

$$\rho_0(y) = \sqrt{\frac{\exp\left[-\left(\frac{\pi}{2n}\right)^2\right]}{n\pi^{\frac{3}{2}}\operatorname{erf}\frac{\pi}{2n}}}\sqrt{y}+\ldots, \qquad y \ll 1. \tag{19b}$$

Obviously $\rho_0(0) = 0$, and the problem is indeed one of class (3).

† Equation (19b) of course could alternatively be obtained by setting $\xi = 0$ in equation (18b).

The real heat flux must now be determined. Using the heat balance condition (15a) with the temperature (17a) the real heat flux in non-dimensional form becomes

$$\frac{Q(\rho, y)}{Q_0} = \frac{2}{m}\frac{\partial \xi}{\partial y} + \frac{1}{2}\left[1 + \frac{1}{n^2}\left(\frac{\partial \xi}{\partial \rho}\right)^2\right]\left\{e^{-(\pi\xi/n)}\left[1 + \operatorname{erf}\left(\frac{\pi}{2n}\sqrt{(y+1)} - \frac{\xi}{\sqrt{(y+1)}}\right)\right]\right.$$
$$\left. + e^{(\pi\xi/n)}\left[1 - \operatorname{erf}\left(\frac{\pi}{2n}\sqrt{(y+1)} + \frac{\xi}{\sqrt{(y+1)}}\right)\right]\right\}\cos \pi\rho. \tag{20a}$$

Substituting for ξ, and expanding for short times, one obtains:

$$\frac{Q}{Q_0}(\rho, y) = \frac{1}{m\sqrt{\pi}} e^{-(\pi/2n)^2} + 1 + \dots \qquad y \ll 1. \tag{20b}$$

Therefore, the problem solved corresponds to the following heating history. The heat flux is $Q_0 \cos \pi\rho$ for all ρ until melting. At $t = t_m$, the real heat flux is still given by $Q_0 \cos \pi\rho$ except at $\rho = 0$ where a jump to the value

$$\left[\frac{1}{m\sqrt{\pi}} e^{-(\pi/2n)^2} + 1\right]$$

occurs. For $t > t_m$, the growth of the melted region is given by equations (18b) and (19b) for the z- and x-directions, respectively. Note that the initial growth rate in the z-direction is in agreement with that predicted by equations (7b), and the initial growth rate in the x-direction is infinite.

2.7. *Problems of class (1)*

Problems of class (1) are mathematically simpler than those of classes (2) or (3) because, although the shape of the solid–liquid interface is still a function of both x and t, the position of the end-points of the interface remains fixed at $x = \pm a$; in other words, the variable $\rho_0(y)$ is absent. In the remainder of this paper, problems of class (1) only are treated: in Section 3 problems in which the melted portion is instantaneously removed are studied, and in Section 4 a type of problem in which both phases are present is taken up.

3. TWO-DIMENSIONAL MELTING OF A FINITE SLAB WITH INSTANTANEOUS REMOVAL OF THE MELT

3.1. *Formulation of the problem*

Consider a solid slab of width $2a$, initially ($t = 0$) at zero temperature, insulated on the sides $x = \pm a$ and $z = l$ (Fig. 1), and let it be subjected to linear heat transfer on the boundary $z = 0$. At some time t_m the surface $z = 0$ reaches the melting temperature, with the further condition, from equation (3d), that

$$\frac{\partial T_N}{\partial t}(x, 0, t_m) > 0, \qquad \text{all } |x| \leq a, \tag{21}$$

so that the problem is of class (1). It is assumed that the melted portion is immediately removed upon formation.

The unknowns $T(x, z, t)$ and $\mathscr{s}(x, t)$ satisfy the following boundary value problem for $t > t_m$: the Fourier heat conduction equation

$$\kappa\left\{\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial z^2}\right\} = \frac{\partial T}{\partial t}; \qquad |x| < a, \quad \mathscr{s}(x, t) < z < l, \quad t \geq t_m, \tag{22a}$$

with the conditions

$$\frac{\partial T}{\partial x}(\pm a, z, t) = 0, \tag{22b}$$

$$\frac{\partial T}{\partial z}(x, l, t) = 0, \tag{22c}$$

$$T(x, \mathscr{s}, t) = T_m, \tag{22d}$$

$$k\left[1+\left(\frac{\partial \mathscr{s}}{\partial x}\right)^2\right]\frac{\partial T}{\partial z}+hT = -Q(x, t)+\rho_S L\frac{\partial \mathscr{s}}{\partial t}; \qquad \text{at } z = \mathscr{s}(x, t), \tag{22e}$$

$$T(x, z, t_m) = T_1(x, z), \tag{22f}$$

$$\mathscr{s}(x, t_m) = 0, \tag{22g}$$

where k, h, κ, ρ_S, L are constants, $Q(x, t)$ is a known flux,† and $T_1(x, z)$ represents the initial condition on the melting problem, that is a known function obtained from the pre-melting solution. The pre-melting solution satisfies (22a) for $0 < z < l$, $0 < t < t_m$, (22b), (22c), with zero initial temperature and linear heat transfer on $z = 0$, which may or may not be the same as that of (22e), with $\partial \mathscr{s}/\partial t = 0$.

Sometimes the assumption is made, in engineering ablation analyses, that there is no heat penetration into the solid, or in other words, there are no thermal gradients in the solid due to external heating. With this assumption the only equations that need be satisfied are (22d, e, g), with the melt line given by

$$\rho_S L\frac{\partial \mathscr{s}}{\partial t} = Q(x, t)+hT_m. \tag{23}$$

For large heat fluxes, such as, for example, those experienced by a body during re-entry, this solution provides a reasonable upper bound.

To obtain a solution to the general problem formulated above, the solid region (which at time $t, t > t_m$, occupies the region $\mathscr{s} < z < l$) is extended so as to occupy the region $0 < z < l$. The temperature field $T_E(x, z, t)$ in this extended region must satisfy the heat conduction equation (22a), for $0 < z < l$, the boundary conditions (22b, c) and the initial condition (22f). Further, an unknown fictitious heat flux $Q^*(x, t)$ is added at the boundary $z = 0$, resulting in the following boundary condition there

$$k\frac{\partial T_E}{\partial z}+hT_E = -[Q(x, t)+Q^*(x, t)] \qquad \text{at } z = 0, \quad t \geq t_m. \tag{24}$$

The functions $\mathscr{s}$ and Q^* are such that the melting conditions (22d, e) are satisfied at $z = \mathscr{s}(x, t)$ under the initial condition (22g).

† The flux is in the z-direction; see [21].

Only problems with symmetry about $x = 0$ will be studied, though it will be evident that removal of this restriction would cause no difficulties. For these problems, then, let

$$T_E(x, z, t) = \sum_{i=0}^{\infty} A_i(z, t) \cos\frac{i\pi x}{a}, \tag{25a}$$

$$T_I(x, z) = \sum_{i=0}^{\infty} B_i(z) \cos\frac{i\pi x}{a}, \tag{25b}$$

$$Q(x, t) = \sum_{i=0}^{\infty} D_i(t) \cos\frac{i\pi x}{a}, \tag{25c}$$

$$\mathscr{J}(x, t) = \sum_{i=0}^{\infty} \bar{E}_i(t) \cos\frac{i\pi x}{a}, \tag{25d}$$

$$Q^*(x, t) = \sum_{i=0}^{\infty} \bar{F}_i(t) \cos\frac{i\pi x}{a}. \tag{25e}$$

Upon substitution of (25a, b, c, e) into equations (22a, b, c, f) and (24), the boundary value problem on $T_E(x, z, t)$ is reduced to a one-dimensional problem for each of the variables $A_i(z, t)$. With the transformation $A_i(z, t) = A_i'(z, t)\exp[-\kappa(i\pi/a)^2 t]$ and the one-dimensional solution for a slab under linear heat-transfer and under initial conditions in terms of B_i, the following temperature field is obtained:

$$\begin{aligned} T_E(x, z, t) = \sum_{i=0}^{\infty} \exp\left[-\kappa\left(\frac{i\pi}{a}\right)^2 t\right]\Bigg\{&\exp\left[\kappa\left(\frac{i\pi}{a}\right)^2 t_m\right]\int_0^l B_i(z')u(z', t-t_m; z)\,\mathrm{d}z' \\ &+\frac{\kappa}{k}\int_{t_m}^{t} [D_i(\tau)+\bar{F}_i(\tau)]\exp\left[\kappa\left(\frac{i\pi}{a}\right)^2\tau\right]u(0, t-\tau; z)\,\mathrm{d}\tau\Bigg\}\cos\frac{i\pi x}{a}, t > t_m, \end{aligned} \tag{26}$$

where $u(z', t-\tau; z)$ is the Green's function or fundamental solution† for the slab under the homogeneous boundary condition

$$k\frac{\partial u}{\partial z}(0, t) + hu(0, t) = 0.$$

This temperature field satisfies all conditions of the original boundary value problem except the conditions (22d, e) on the moving boundary, and the initial condition (22g). These equations take the form, for $t > t_m$,

$$\begin{aligned} \sum_{i=0}^{\infty} \exp\left[-\kappa\left(\frac{i\pi}{a}\right)^2 t\right]\Bigg\{&\exp\left[\kappa\left(\frac{i\pi}{a}\right)^2 t_m\right]\int_0^l B_i(z')u(z', t-t_m; \mathscr{J})\,\mathrm{d}z' \\ &+\frac{\kappa}{k}\int_{t_m}^{t} [D_i(\tau)+\bar{F}_i(\tau)]\exp\left[\kappa\left(\frac{i\pi}{a}\right)^2\tau\right]u(0, t-\tau; \mathscr{J})\,\mathrm{d}\tau\Bigg\}\cos\frac{i\pi x}{a} = T_m. \end{aligned} \tag{27a}$$

† When u satisfies this particular boundary condition it is sometimes referred to as the Robin's function.

$$k\left[1+\left(\frac{\partial \mathscr{s}}{\partial x}\right)^2\right]\sum_{i=0}^{\infty}\exp\left[-\kappa\left(\frac{i\pi}{a}\right)^2 t\right]\left\{\exp\left[\kappa\left(\frac{i\pi}{a}\right)^2 t_m\right]\int_0^l B_i(z')\frac{\partial u}{\partial z}(z', t-t_m; \mathscr{s})\,dz'\right.$$

$$\left.+\frac{\kappa}{k}\int_{t_m}^{t}[D_i(\tau)+\bar{F}_i(\tau)]\exp\left[\kappa\left(\frac{i\pi}{a}\right)^2\tau\right]\frac{\partial u}{\partial z}(0, t-\tau; \mathscr{s})\,d\tau\right\}\cos\frac{i\pi x}{a} \tag{27b}$$

$$= -\sum_{i=0}^{\infty} D_i(t)\cos\frac{i\pi x}{a}+\rho_S L\frac{\partial \mathscr{s}}{\partial t}-hT_m,$$

and

$$\bar{E}_i(t_m) = 0, \qquad i = 0, 1, 2, \ldots. \tag{27c}$$

Where the variable $\mathscr{s}$ appears in these equations its series form (25d) is understood.

Equations (37a, b) are two coupled, non-linear, integro-differential equations in the unknowns $\bar{F}_i(t)$, $\bar{E}_i(t)$; once $\bar{F}_i(t)$, $\bar{E}_i(t)$ are found, the solution is complete since all equations of the original boundary value problem are satisfied. The temperature field is then obtained directly from equation (26). $T_E(x, z, t)$, of course, represents a temperature field in the entire slab $0 < z < l$; however, only the temperature field for $z > \mathscr{s}$ has physical meaning, and corresponds to the temperature field in the solid region.

3.2. *Example for Section 3.1*

A specific example will now be considered, in which $l = \infty$, $h = 0$, and the strip is heated as follows (see inset of Fig. 2):

$$Q(x, t) = \begin{cases} Q_0 & 0 < t < t_m \\ Q_0\left[1+C_0\left(1+C_1\cos\dfrac{\pi x}{a}\right)\right] & t \geq t_m \end{cases} \tag{28a}$$

where Q_0, C_0 and C_1 are constants satisfying the inequalities

$$\begin{aligned} C_0 &\geq 0, \\ |C_1| &\leq 1, \end{aligned} \tag{28b}$$

which insure that the problem is one of class (1).†

The pre-melting solution is

$$T(x, z, t) = \frac{2Q_0\sqrt{(\kappa t)}}{k}\operatorname{ierfc}\frac{z}{2\sqrt{(\kappa t)}}, \qquad 0 < t \leq t_m, \tag{29}$$

from which

$$T_I(x, z) = \frac{2Q_0\sqrt{(\kappa t_m)}}{k}\operatorname{ierfc}\frac{z}{2\sqrt{(\kappa t_m)}}; \qquad T_m = \frac{2Q_0\sqrt{(\kappa t_m)}}{\sqrt{(\pi)}k}. \tag{30}$$

The Green's function is

$$u(z', t-\tau; z) = \frac{1}{2\sqrt{[\pi\kappa(t-\tau)]}}\left\{\exp\left[-\frac{(z-z')^2}{4\kappa(t-\tau)}\right]+\exp\left[-\frac{(z+z')^2}{4\kappa(t-\tau)}\right]\right\} \tag{31}$$

† This follows immediately from (21) and the one-dimensional initial behavior as in the problem of Section 2.2.

and the general equations (27a, b) become:

$$\frac{\sqrt{\kappa}}{k\sqrt{\pi}}\sum_{i=0}^{\infty}\int_0^{t-t_m}\frac{\bar{F}_i(t-\tau)}{\sqrt{\tau}}\exp\left\{-\left[\kappa\left(\frac{i\pi}{a}\right)^2\tau+\frac{\mathscr{s}^2}{4\kappa\tau}\right]\right\}\mathrm{d}\tau\cos\frac{i\pi x}{a} = T_m-\frac{2Q_0\sqrt{(\kappa t)}}{k}\operatorname{ierfc}\frac{\mathscr{s}}{2\sqrt{(\kappa t)}}$$
$$-\frac{2Q_0C_0}{k}\sqrt{[\kappa(t-t_m)]}\operatorname{ierfc}\frac{\mathscr{s}}{2\sqrt{[\kappa(t-t_m)]}}$$
$$-\frac{Q_0C_0C_1a}{2\pi k}\left\{\exp(-\pi\mathscr{s}/a)\left[1+\operatorname{erf}\left(\frac{\pi}{a}\sqrt{[\kappa(t-t_m)]}-\frac{\mathscr{s}}{2\sqrt{[\kappa(t-t_m)]}}\right)\right]\right. \tag{32a}$$
$$\left.-\exp(\pi\mathscr{s}/a)\left[1-\operatorname{erf}\left(\frac{\pi}{a}\sqrt{[\kappa(t-t_m)]}+\frac{\mathscr{s}}{2\sqrt{[\kappa(t-t_m)]}}\right)\right]\right\}\cos\frac{\pi x}{a},$$

$$\frac{\mathscr{s}}{2\sqrt{(\pi\kappa)}}\left[1+\left(\frac{\partial\mathscr{s}}{\partial x}\right)^2\right]\sum_{i=0}^{\infty}\int_0^{t-t_m}\frac{\bar{F}_i(t-\tau)}{(\tau)^{\frac{3}{2}}}\exp-\left[\kappa\left(\frac{i\pi}{a}\right)^2\tau+\frac{\mathscr{s}^2}{4\kappa\tau}\right]\mathrm{d}\tau\cos\frac{i\pi x}{a}$$
$$= -\rho_S L\frac{\partial\mathscr{s}}{\partial t}+Q_0\operatorname{erf}\frac{\mathscr{s}}{2\sqrt{(\kappa t)}}+Q_0C_0\operatorname{erf}\frac{\mathscr{s}}{2\sqrt{[\kappa(t-t_m)]}}$$
$$+Q_0C_0C_1\left[1-\tfrac{1}{2}\left\{\exp(-\pi\mathscr{s}/a)\left(1+\operatorname{erf}\left[\frac{\pi}{a}\sqrt{[\kappa(t-t_m)]}-\frac{\mathscr{s}}{2\sqrt{[\kappa(t-t_m)]}}\right]\right)\right.\right.$$
$$\left.\left.+\exp(\pi\mathscr{s}/a)\left(1-\operatorname{erf}\left[\frac{\pi}{a}\sqrt{[\kappa(t-t_m)]}+\frac{\mathscr{s}}{2\sqrt{[\kappa(t-t_m)]}}\right]\right)\right\}\right]\cos\frac{\pi x}{a} \tag{32b}$$
$$-\left(\frac{\partial\mathscr{s}}{\partial\rho}\right)^2Q_0\left[\operatorname{erfc}\frac{\mathscr{s}}{2\sqrt{(\kappa t)}}+C_0\operatorname{erfc}\frac{\mathscr{s}}{2\sqrt{[\kappa(t-t_m)]}}+\right.$$
$$+\frac{C_0C_1}{2}\left\{\exp(-\pi\mathscr{s}/a)\left(1+\operatorname{erf}\left[\frac{\pi}{a}\sqrt{[\kappa(t-t_m)]}-\frac{\mathscr{s}}{2\sqrt{[\kappa(t-t_m)]}}\right]\right)\right.$$
$$\left.\left.+\exp(\pi\mathscr{s}/a)\left(1-\operatorname{erf}\left[\frac{\pi}{a}\sqrt{[\kappa(t-t_m)]}+\frac{\mathscr{s}}{2\sqrt{[\kappa(t-t_m)]}}\right]\right)\right\}\cos\pi\rho\right].$$

The following non-dimensional variables and parameters (cf. [11]) are introduced:

$$y=\frac{t-t_m}{t_m};\qquad \rho=\frac{x}{a};\qquad \zeta=\frac{z}{a};\qquad F_i(y)=\frac{\bar{F}_i}{Q_0}; \tag{33}$$

$$\xi=\frac{\mathscr{s}}{2\sqrt{(\kappa t_m)}};\qquad n=\frac{a}{2\sqrt{(\kappa t_m)}};\qquad m=\frac{Q_0\sqrt{t_m}}{\rho_S\sqrt{(\kappa)}L};\qquad E_i(y)=\frac{\bar{E}_i}{2\sqrt{(\kappa t_m)}}.$$

Equations (32a, b) in non-dimensional form are then:

$$\sum_{i=0}^{\infty}\int_0^y \frac{F_i(y-y_1)}{\sqrt{y_1}}\exp\left\{-\left[\left(\frac{i\pi}{2n}\right)^2 y_1+\frac{\xi^2}{y_1}\right]\right\} dy_1 \cos i\pi\rho$$
$$= 2\left\{1-\sqrt{[\pi(y+1)]}\,\text{ierfc}\,\frac{\xi}{\sqrt{(y+1)}}\right\}-2C_0\sqrt{(\pi y)}\,\text{ierfc}\,\frac{\xi}{\sqrt{y}}$$
$$-\frac{C_0C_1n}{\sqrt{\pi}}\left\{e^{-(\pi\xi/n)}\left(1+\text{erf}\left[\frac{\pi}{2n}\sqrt{y}-\frac{\xi}{\sqrt{y}}\right]\right)\right.$$
$$\left.-e^{\pi\xi/n}\left(1-\text{erf}\left[\frac{\pi}{2n}\sqrt{y}+\frac{\xi}{\sqrt{y}}\right]\right)\right\}\cos\pi\rho, \tag{34a}$$

$$\left[1+\frac{1}{n^2}\left(\frac{\partial\xi}{\partial\rho}\right)^2\right]\frac{\xi}{\sqrt{\pi}}\sum_{i=0}^{\infty}\int_0^y \frac{F_i(y-y_1)}{y_1^{3/2}}\exp\left\{-\left[\left(\frac{i\pi}{2n}\right)^2 y_1+\frac{\xi^2}{y_1}\right]\right\}dy_1\cos i\pi\rho$$
$$= -\frac{2}{m}\frac{\partial\xi}{\partial y}+\text{erf}\,\frac{\xi}{\sqrt{(y+1)}}+C_0\,\text{erf}\,\frac{\xi}{\sqrt{y}}+C_0C_1\left[1-\frac{1}{2}\left\{e^{-(\pi\xi/n)}\left(1+\text{erf}\left[\frac{\pi}{2n}\sqrt{y}-\frac{\xi}{\sqrt{y}}\right]\right)\right.\right.$$
$$\left.\left.+e^{\pi\xi/n}\left(1-\text{erf}\left[\frac{\pi}{2n}\sqrt{y}+\frac{\xi}{\sqrt{y}}\right]\right)\right\}\right]\cos\pi\rho-\frac{1}{n^2}\left(\frac{\partial\xi}{\partial\rho}\right)^2\left[\text{erfc}\,\frac{\xi}{\sqrt{(y+1)}}+C_0\,\text{erfc}\,\frac{\xi}{\sqrt{y}}\right.$$
$$\left.+\frac{C_0C_1}{2}\left\{e^{-(\pi\xi/n)}\left(1+\text{erf}\left[\frac{\pi}{2n}\sqrt{y}-\frac{\xi}{\sqrt{y}}\right]\right)+e^{\pi\xi/n}\left(1-\text{erf}\left[\frac{\pi}{2n}\sqrt{y}+\frac{\xi}{\sqrt{y}}\right]\right)\right\}\cos\pi\rho\right], \tag{34b}$$

for $y \geq 0$, $|\rho| \leq 1$, $\xi(\rho, y) \geq 0$, and

$$E_i(0) = 0 \qquad i = 0, 1, 2, \ldots. \tag{34c}$$

A very short time solution can be found as follows. It was seen in equation (7b), that for a finite jump in flux at $t = t_m$ the initial melting rate is finite; hence $\xi \propto y$ and

$$\lim_{y\to 0}\frac{\xi}{\sqrt{y}} \to 0.$$

Therefore the exponential appearing in the integrands of the above equations, is, for small y,

$$\exp\left\{-\left[\left(\frac{i\pi}{2n}\right)^2 y_1+\frac{\xi^2}{y_1}\right]\right\} \cong 1$$

except for, $y_1 \to 0$, i.e. very near the lower limit of integration, where it goes rapidly to 0. Therefore, expanding the right-hand side of (34a) for small y and approximating the exponential by 1, the following short time integral equation is obtained:†

$$\sum_{i=0}^{\infty}\int_0^y \frac{F_i(y-y_1)}{\sqrt{y_1}}dy_1\cos i\pi\rho = -2C_0\sqrt{(y)}(1+C_1\cos\pi\rho+\ldots), \qquad y \ll 1. \tag{35a}$$

† The similarity of this reasoning with that of [11] is evident. An alternative procedure consists of a development analogous to that of [21].

This is a particular type of Abel's integral equation [18] whose solution is

$$\begin{aligned} F_0(y) &= -C_0 \\ F_1(y) &= -C_0C_1 \qquad y \ll 1 \\ F_i(y) &= 0 \qquad i = 2, 3, 4, \ldots . \end{aligned} \tag{35b}$$

Substitution into equation (34b) and expansion of the right-hand side for small y leads to the following differential equation:

$$-C_0\left[1+C_1\cos\pi\rho+\ldots\right] = -\frac{2}{m}\sum_{i=0}^{\infty}\frac{\partial E_i(y)}{\partial y}\cos i\pi\rho+\ldots \qquad y \ll 1 \tag{36a}$$

whose solution, satisfying the initial condition on $E_i(y)$, is

$$\begin{aligned} E_0(y) &= \frac{m}{2}C_0y, \\ E_1(y) &= \frac{m}{2}C_0C_1y, \qquad y \ll 1 \\ E_i(y) &= 0 \qquad i = 2, 3, 4, \ldots . \end{aligned} \tag{36b}$$

The above solution provides a starting response, valid for $y \ll 1$. To extend the range of validity of the solution in the time domain, expansions of the following form are considered:

$$\frac{Q^*}{Q_0}(\rho, y) = (-C_0 - C_0C_1\cos\pi\rho) + \sum_{i=0}^{\infty}(a_{i1}\sqrt{y}+a_{i2}y+a_{i3}y^{\frac{3}{2}}+\ldots)\cos i\pi\rho, \tag{37a}$$

$$\xi(\rho, y) = \left(\frac{mC_0}{2}y+\frac{mC_0C_1}{2}y\cos\pi\rho\right) + \sum_{i=0}^{\infty}(b_{i1}y^{\frac{3}{2}}+b_{i2}y^2+b_{i3}y^{\frac{5}{2}}+\ldots)\cos i\pi\rho. \tag{37b}$$

These series can now be substituted into equations (34a, b) and the resulting expressions expanded in powers of y. A solution can then be constructed by equating coefficients of like cosines and like powers of y. After considerable effort,† the following solution for the fictitious heat flux and melt thickness results:

$$\begin{aligned} \frac{Q^*}{Q_0}(\rho, y) = {} & \left[-C_0+\left(-\frac{2}{\pi}+\frac{2mC_0}{\sqrt{\pi}}\right)\sqrt{y}+\left(\frac{m}{\sqrt{\pi}}-m^2C_0-\frac{3mC_0}{2\sqrt{\pi}}+\frac{3m^2C_0^2}{2}+\frac{3m^2C_0^2C_1^2}{4}\right)y+\ldots\right] \\ & +\left[-C_0C_1+\frac{2mC_0C_1}{\sqrt{\pi}}\sqrt{y}+\left(-m^2C_0C_1-\frac{3mC_0C_1}{2\sqrt{\pi}}+3m^2C_0^2C_1\right)y+\ldots\right]\cos\pi\rho \\ & +\left[\frac{3m^2C_0^2C_1^2}{4}y+\ldots\right]\cos 2\pi\rho+\ldots, \end{aligned} \tag{37c}$$

† A list of the principal integrals arising in the process of obtaining this solution is given in [21].

$$\xi(\rho, y) = \left[\frac{mC_0}{2}y + \left(\frac{2m}{3\pi} - \frac{2m^2C_0}{3\sqrt{\pi}}\right)y^{\frac{3}{2}} + \left(-\frac{m^2}{4\sqrt{\pi}} + \frac{3m^2C_0}{8\sqrt{\pi}} - \frac{m^3C_0^2}{8}\right.\right.$$
$$\left.\left. + \frac{m^3C_0}{4} - \frac{3m^3C_0^2C_1^2}{16} - \frac{m^3C_0^2C_1}{8}\right)y^2 + \dots\right] + \left[\frac{mC_0C_1}{2}y - \frac{2m^2C_0C_1}{3\sqrt{\pi}}y^{\frac{3}{2}}\right.$$
$$\left. + \left(\frac{3m^2C_0C_1}{8\sqrt{\pi}} - \frac{m^3C_0^2C_1}{2} + \frac{m^3C_0^2}{4} + \frac{m^3C_0C_1}{4}\right)y^2 + \dots\right]\cos \pi\rho \tag{37d}$$
$$+ \left[\left(-\frac{3m^3C_0^2C_1^2}{16} + \frac{m^3C_0^2C_1}{8}\right)y^2 + \dots\right]\cos 2\pi\rho + \dots.$$

The unknown temperature field can be obtained from equation (26) (or, more conveniently, it can be deduced from (34a) as

$$\frac{T_E}{T_m}(\rho, \zeta, y) = \frac{1}{2}\sum_{i=0}^{\infty}\int_0^y \frac{F_i(y-y_1)}{\sqrt{y_1}}\exp\left\{-\left[\left(\frac{i\pi}{2n}\right)^2 y_1 + \frac{n^2\zeta^2}{y_1}\right]\right\} \mathrm{d}y_1 \cos i\pi\rho$$
$$+ \sqrt{[\pi(y+1)]}\,\mathrm{ierfc}\frac{n\zeta}{\sqrt{(y+1)}} + C_0\sqrt{(\pi y)}\,\mathrm{ierfc}\frac{n\zeta}{\sqrt{y}} \tag{38a}$$
$$+ \frac{C_0C_1 n}{2\sqrt{\pi}}\left[e^{-\pi\zeta}\left(1 + \mathrm{erf}\left[\frac{\pi}{2n}\sqrt{y} - \frac{n\zeta}{\sqrt{y}}\right]\right) - e^{\pi\zeta}\left(1 - \mathrm{erf}\left[\frac{\pi}{2n}\sqrt{y} + \frac{n\zeta}{\sqrt{y}}\right]\right)\right]\cos \pi\rho,$$

and with the fictitious heat flux $F_i(y)$ it becomes:

$$\frac{T_E}{T_m}(\rho, \zeta, y) = \sqrt{[\pi(y+1)]}\,\mathrm{ierfc}\frac{n\zeta}{\sqrt{(y+1)}} + [-2 + 2\sqrt{(\pi)}mC_0]y\,\mathrm{i}^2\mathrm{erfc}\frac{n\zeta}{\sqrt{y}}$$
$$+ [4m - 4\sqrt{(\pi)}m^2C_0 - 6mC_0 + 6\sqrt{(\pi)}m^2C_0^2 + 3\sqrt{(\pi)}m^2C_0^2C_1^2]y^{\frac{3}{2}}\,\mathrm{i}^3\mathrm{erfc}\frac{n\zeta}{\sqrt{y}} + \dots$$
$$+ \left\{2\sqrt{(\pi)}mC_0C_1 y\,\mathrm{i}^2\mathrm{erfc}\frac{n\zeta}{\sqrt{y}} + \left[-4\sqrt{(\pi)}m^2C_0C_1 - 6mC_0C_1 + 12\sqrt{(\pi)}m^2C_0^2C_1\right.\right. \tag{38b}$$
$$\left.\left. - \frac{\pi^{\frac{3}{2}}mC_0C_1}{n}\zeta\right]y^{\frac{3}{2}}\,\mathrm{i}^3\mathrm{erfc}\frac{n\zeta}{\sqrt{y}} + \dots\right\}\cos \pi\rho$$
$$+ \left[3\sqrt{(\pi)}m^2C_0^2C_1^2 y^{\frac{3}{2}}\,\mathrm{i}^3\mathrm{erfc}\frac{n\zeta}{\sqrt{y}} + \dots\right]\cos 2\pi\rho + \dots.$$

This temperature field has physical meaning, of course, only for $\zeta \geq \xi$.

The solution of the problem is now complete. Curves showing the variation of the melting thickness with x and t are presented in Fig. 2 (on p. 212). The temperature along the axis of symmetry is plotted for several values of time in Fig. 3, and the temperature field at a fixed time is shown in Fig. 4. The dotted portions of the curves of Figs. 3 and 4 represent the temperature field in the fictitious region.

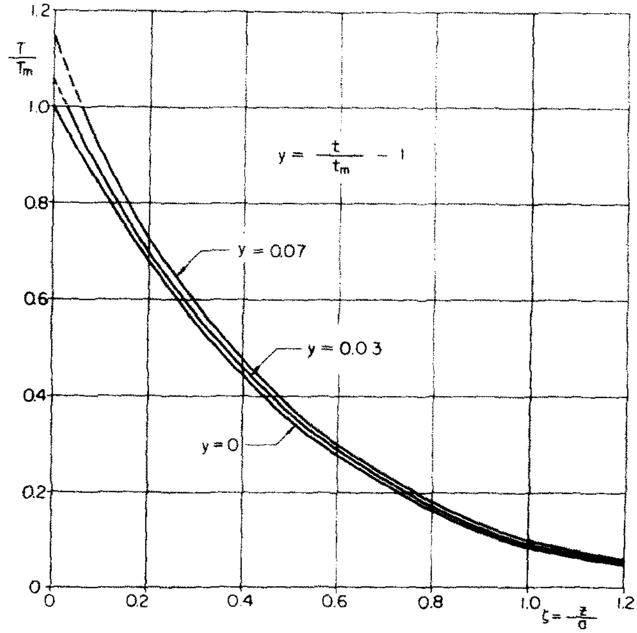

FIG. 3. A dimensionless plot of the temperature field for the melting problem vs. z/a along the axis of symmetry for various times and for $m = n = C_0 = C_1 = 1$.

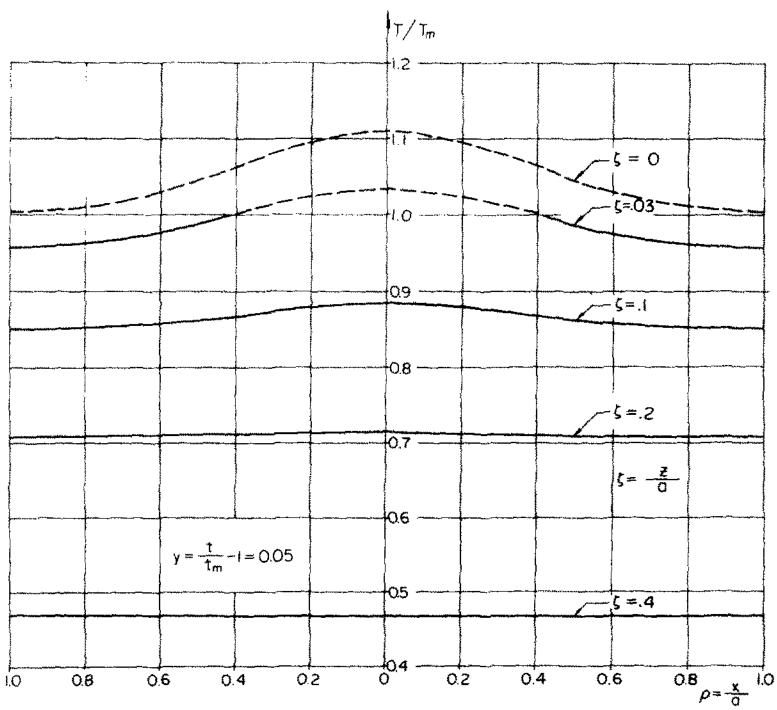

FIG. 4. A dimensionless plot of the temperature field for the melting problem vs. x/a for $y = 0{\cdot}05$, at various depths in the strip and for $m = n = C_0 = C_1 = 1$.

3.3. *Discussion of the solution of Section 3.2*

The following remarks concerning the solution presented in the preceding section are of interest:

(1) Two one-dimensional solutions of interest can be obtained directly from the two-dimensional results. For $C_0 = 0$, the solution for the problem of a half space under the action of a continuous heat flux Q_0 is obtained. The resulting melt thickness is then:†

$$\xi(y) = \frac{2m}{3\pi}y^{\frac{3}{2}} - \frac{m^2}{4\sqrt{\pi}}y^2 - \frac{4m}{15\pi}\left(\tfrac{1}{2} - m^2 - \frac{16m}{3\pi^{\frac{3}{2}}}\right) + \dots \tag{39}$$

and checks identically with that given in [11]. For $C_1 = 0$, the half space is subjected to the flux Q_0 until melting occurs, at which time an additional flux $Q_1 = Q_0C_0$ is added. The melting thickness here is given by

$$\xi(y) = \frac{mC_0}{2}y + \left(\frac{2m}{3\pi} - \frac{2m^2C_0}{3\sqrt{\pi}}\right)y^{\frac{3}{2}} + \left(-\frac{m^2}{4\sqrt{\pi}} + \frac{3m^2C_0}{8\sqrt{\pi}} - \frac{m^3C_0^2}{8} + \frac{m^3C_0}{4}\right)y + \dots \tag{40}$$

and was discussed in Section 2.3.

(2) It can be noticed that, although the applied heat input is described using only two terms of a Fourier series, all terms appear in the solution. However, for very short times, an essentially one-dimensional behavior prevails, with the temperature and the melt thickness proportional to the heat input. The power of the leading term in y in the coefficients of $\cos 2\pi\rho$ is higher than that in the coefficient of the first two terms of the series, and it may be conjectured that the leading power of y will increase with the successive cosine terms. The validity of this conjecture was not tested, though it was shown to hold for the next term for the special case in which $C_1 = 1$. In this case the next term in the solution for the fictitious heat input Q^* is

$$\begin{aligned} y^{\frac{3}{2}}\Bigg\{&\left[\frac{4}{3\pi}\left(\frac{1}{2} - m^2 - \frac{16m}{3\pi^{\frac{3}{2}}}\right) + \frac{4m^3C_0}{3\sqrt{\pi}} + m^2C_0\left(\frac{14}{3\pi} + \frac{128}{9\pi^2}\right) - m^3C_0^2\left(\frac{32}{3\pi^{\frac{3}{2}}} + \frac{15}{3\sqrt{\pi}}\right) - \frac{6m^2C_0^2}{\pi}\right.\\ &\left. - m^3C_0^3\left(\frac{8}{\sqrt{\pi}} + \frac{4}{3\pi^{\frac{3}{2}}}\right)\right] + \left[\frac{\pi^{\frac{3}{2}}}{6n^2}mC_0 - \frac{8m^2C_0^2}{\pi} + \frac{4m^3C_0}{3\sqrt{\pi}} + \frac{10m^3C_0^3}{\sqrt{\pi}}\right.\\ &\left. + m^2C_0\left(\frac{128}{9\pi^2} + \frac{14}{3\pi}\right) - m^3C_0^2\left(\frac{128}{9\pi^{\frac{3}{2}}} + \frac{20}{3\sqrt{\pi}}\right)\right] \cos \pi\rho\\ &+ \left[-\frac{2}{\pi}m^2C_0^2 - m^3C_0^2\left(\frac{32}{9\pi^{\frac{3}{2}}} + \frac{5}{3\sqrt{\pi}}\right) + m^3C_0^3\left(\frac{16}{3\sqrt{\pi}} - \frac{4}{3\pi^{\frac{3}{2}}}\right)\right] \cos 2\pi\rho + \left[\frac{2m^3C_0^3}{3\sqrt{\pi}}\right] \cos 3\pi\rho\Bigg\} \end{aligned}$$

while the next term in the solution for the dimensionless melt thickness ξ is

$$\begin{aligned} y^{\frac{5}{2}}\Bigg\{&\left[-\frac{4m}{15\pi}\left(\frac{1}{2} - m^2 - \frac{16m}{3\pi^{\frac{3}{2}}}\right) - m^3C_0\left(\frac{4}{15\pi} + \frac{128}{45\pi^2}\right) + \frac{9m^3C_0^2}{20\pi} - \frac{4m^4C_0}{15\sqrt{\pi}} + \frac{32m^4C_0^2}{3\pi^{\frac{3}{2}}}\right.\\ &\left. + m^4C_0^3\left(\frac{57}{20\sqrt{\pi}} + \frac{4}{15\pi^{\frac{3}{2}}}\right)\right] + \left[-\frac{\pi^{\frac{3}{2}}}{30n^2}m^2C_0 - m^3C_0\left(\frac{4}{5\pi} + \frac{128}{45\pi^2}\right)\right. \end{aligned}$$

† The last term is obtained from the solution given in (2) below.

$$+\frac{3m^3C_0^2}{5\pi}-\frac{4m^4C_0}{15\sqrt{\pi}}+m^4C_0^2\left(\frac{8}{15\sqrt{\pi}}+\frac{128}{45\pi^{\frac{3}{2}}}\right)-\frac{m^4C_0^3}{8\sqrt{\pi}}\Bigg]\cos\pi\rho$$

$$+\left[\frac{3m^3C_0^2}{20\pi}+\frac{32m^4C_0^2}{45\pi^{\frac{3}{2}}}-m^4C_0^3\left(\frac{19}{60\sqrt{\pi}}-\frac{4}{15\pi^{\frac{3}{2}}}\right)\right]\cos 2\pi\rho+\left[-\frac{m^4C_0^3}{120\sqrt{\pi}}\right]\cos 3\pi\rho\Bigg\}.$$

(3) The numerical results shown in Figs. 2, 3 and 4 were obtained using all of the terms of the series (37b, c). In all cases, however, the contribution of the last terms of both the series in t and x was small compared to that of the preceding terms.

(4) Examination of Fig. 4 shows that the temperature distribution is markedly two-dimensional near the heated surface, but becomes increasingly uniform as one proceeds into the interior of the strip. This is to be expected since the pre-melting solution is one-dimensional.

4. TWO-DIMENSIONAL SOLIDIFICATION OF A SLAB WITH ZERO SUPERHEAT

4.1. *Formulation of the problem*

The slab of Fig. 1 is assumed to be initially liquid and uniformly at the melting temperature T_m; it is subjected to linear heat transfer on the boundary $z = 0$ for $t > 0$ such that

$$\frac{\partial T_N}{\partial t}(x, 0, 0) < 0 \qquad \text{all } |x| \le a \tag{41}$$

insuring that the problem belongs to class (1). Since there is no superheat† solidification begins immediately (at $t = 0$), and the temperature in the liquid, of course, always remaining at T_m.

The temperature field in the solid, $T(x, z, t)$, and the solid thickness $\mathscr{S}(x, t)$, measured in the z-direction, satisfy the following boundary value problem for $t > 0$: the Fourier heat conduction equation

$$\kappa\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial z^2}\right) = \frac{\partial T}{\partial t} \qquad |x| < a,\ 0 < z < \mathscr{S}(x, t), \tag{42a}$$

and the conditions

$$\frac{\partial T}{\partial x}(\pm a, z, t) = 0, \tag{42b}$$

$$k\frac{\partial T}{\partial z}(x, 0, t)+hT(x, 0, t) = Q(x, t), \tag{42c}$$

$$T(x, \mathscr{S}, t) = T_m, \tag{42d}$$

$$k\left[1+\left(\frac{\partial \mathscr{S}}{\partial x}\right)^2\right]\frac{\partial T}{\partial z} = \rho_S L\frac{\partial \mathscr{S}}{\partial t} \qquad \text{at } z = \mathscr{S}(x, t), \tag{42e}$$

† If the superheat were not zero, there would be thermal gradients in the liquid as well as the solid; in such a case the present analysis can be extended in a manner analogous to that of [11]. For a physical discussion of the assumption of zero superheat and freezing problems in general see for example [19].

$$T(x,0,0) = T_m, \tag{42f}$$

$$\mathscr{s}(x,0) = 0, \tag{42g}$$

where κ, k, h, ρ_S and L are constants, and $Q(x,t)$ is a known flux (heat is extracted from the solid). This formulation is valid for $\mathscr{s}(x,t) < l$.

The solution will be again obtained using the concept of a fictitious body, in which the real body is embedded, under a fictitious heating condition. However, the approach here is different from that of Section 3 because of the presence of both the liquid and solid regions and because the boundary $z = 0$ is now a real boundary where condition (42c) must be satisfied. The solid region can, however, be extended to† $z = \infty$, thus again replacing it by a fictitious region of constant geometry, and a fictitious initial temperature distribution $T^*(x,z)$ can be introduced such that the interface conditions on $z = \mathscr{s}(x,t)$ are again satisfied. $T_E(x,z,t)$, the temperature field in this extended region, satisfies equation (42a) for $z > 0$, and (42b, c). It must further satisfy the condition

$$T_E(x,z,0) = T^*(x,z), \tag{43a}$$

where, from (42f),

$$T^*(x,0) = T_m. \tag{43b}$$

Considering symmetry about the z-axis, we may introduce the following Fourier series:

$$T_E(x,z,t) = \sum_{i=0}^{\infty} A_i(z,t)\cos\frac{i\pi x}{a}, \tag{44a}$$

$$T^*(x,z) = \sum_{i=0}^{\infty} \bar{B}_i(z)\cos\frac{i\pi x}{a}, \tag{44b}$$

$$Q(x,t) = \sum_{i=0}^{\infty} D_i(t)\cos\frac{i\pi x}{a}, \tag{44c}$$

$$\mathscr{s}(x,t) = \sum_{i=0}^{\infty} \bar{E}_i(t)\cos\frac{i\pi x}{a}. \tag{44d}$$

Upon substitution of (44a, b, c) into equations (42a), (42b, c), (43a) and (43b), the boundary value problem on T_E is reduced to a one-dimensional problem on each of the functions $A_i(z,t)$. With the transformation

$$A_i(z,t) = A_i'(z,t)\exp\left[-\kappa\left(\frac{i\pi}{a}\right)^2 t\right]$$

the following temperature field is obtained

$$T_E(x,z,t) = \sum_{i=0}^{\infty}\exp\left[-\kappa\left(\frac{i\pi}{a}\right)^2 t\right]\left[\int_0^{\infty}\bar{B}_i(z')u(z',t;z)\,\mathrm{d}z' + \frac{\kappa}{k}\int_0^t D_i(\tau)\exp\left[\kappa\left(\frac{i\pi}{a}\right)^2\tau\right]u(0,t-\tau;z)\,\mathrm{d}\tau\right]\cos\frac{i\pi x}{a}, \tag{45}$$

† The length l^* of the extended solid may be chosen at will, provided that $l^* \geq l$, as long as $\mathscr{s} < l$; cf. [11]. In this case $l^* = \infty$ is the simplest choice.

where $u(z', t-\tau; z)$ is the Robin's function for the half space $z > 0$. Condition (43b) to be satisfied by the fictitious initial temperature distribution becomes

$$\bar{B}_0(0) = T_m \qquad \bar{B}_i(0) = 0 \qquad i = 1, 2, \ldots. \tag{46}$$

The temperature field (45), with conditions (46), now satisfies all the conditions of the original boundary value problem with the exception of (42d, e, g). Upon substitution of (45), these equations assume the following form

$$\sum_{i=0}^{\infty} \exp\left[-\kappa\left(\frac{i\pi}{a}\right)^2 t\right]\left\{\int_0^{\infty} \bar{B}_i(z')u(z', t; \vartheta)\,dz' + \frac{\kappa}{k}\int_0^t D_i(\tau)\exp\left[\kappa\left(\frac{i\pi}{a}\right)^2\tau\right]u(0, t-\tau; \vartheta)\,d\tau\right\}\cos\frac{i\pi x}{a} = T_m, \tag{47a}$$

$$k\left[1+\left(\frac{\partial\vartheta}{\partial x}\right)^2\right]\sum_{i=0}^{\infty} \exp\left[-\kappa\left(\frac{i\pi}{a}\right)^2 t\right]\left\{\int_0^{\infty} \bar{B}_i(z')\frac{\partial u}{\partial z}(z', t; \vartheta)\,dz' + \frac{\kappa}{k}\int_0^t D_i(\tau)\exp\left[\kappa\left(\frac{i\pi}{a}\right)^2\tau\right]\frac{\partial u}{\partial z}(0, t-\tau, \vartheta)\,d\tau\right\}\cos\frac{i\pi x}{a} = \rho_S L\frac{\partial\vartheta}{\partial t}, \tag{47b}$$

and, with (44d)

$$\bar{E}_i(0) = 0 \qquad i = 0, 1, 2, \ldots. \tag{47c}$$

Equation (47a) requires that the temperature along the melt line $z = \vartheta(x, t)$ be equal to the melting temperature T_m, while equation (47b) is the heat-balance condition on $z = \vartheta(x, t)$. The determination of the functions $\bar{B}_i(z)$, $\bar{E}_i(t)$ which satisfy equations (46) and (47) completes the solution to the problem since all conditions of the original boundary value problem are then satisfied. The temperature field is then obtained directly by substituting the quantities $\bar{B}_i(z)$ into (45); of course, the temperature field as obtained has physical meaning only for $0 \leq z \leq \vartheta(x, t)$.

As a one-dimensional special case of the above formulation, the problem of Neumann can be considered. In that problem [2] the temperature at $z = 0$ is held at $T = 0$ where $T_m > 0$; alternatively, a heat input

$$Q(t) = -\frac{kT_m}{\operatorname{erf}\lambda\sqrt{(\pi\kappa t)}}; \qquad \lambda e^{\lambda^2}\operatorname{erf}\lambda = \frac{kT_m}{\kappa\rho_S L\sqrt{\pi}} \tag{48}$$

may be prescribed there. Let $D_0(t) = Q(t)$ and $D_i = 0$, $i > 0$ in equations (47a, b); with $\bar{B}_i(z) = 0$, $i > 0$, the result is

$$\frac{1}{2\sqrt{(\pi\kappa t)}}\int_0^{\infty}\bar{B}_0(z')\left\{\exp\left[-\frac{(\vartheta-z')^2}{4\kappa t}\right]+\exp\left[-\frac{(\vartheta+z')^2}{4\kappa t}\right]\right\}dz' - \frac{T_m}{\operatorname{erf}\lambda}\operatorname{erfc}\frac{\vartheta}{2\sqrt{(\kappa t)}} = T_m, \tag{49a}$$

$$-\frac{k}{4\sqrt{\pi\kappa^{3}t^{3}}}\int_0^{\infty}\bar{B}_0(z')\left\{(\vartheta-z')\exp\left[-\frac{(\vartheta-z')^2}{4\kappa t}\right]+(\vartheta+z')\exp\left[-\frac{(\vartheta+z')^2}{4\kappa t}\right]\right\}dz' + \frac{kT_m}{\sqrt{(\pi\kappa t)}\operatorname{erf}\lambda} = \rho_S L\frac{d\vartheta}{dt}. \tag{49b}$$

The solution to equations (49a, b) can be verified to be

$$\begin{aligned} \bar{B}_0(z') &= T_m/\operatorname{erf}\lambda, \\ \mathscr{S}(t) &= 2\lambda\sqrt{(\kappa t)}. \end{aligned} \tag{50}$$

The solid thickness $\mathscr{S}(t)$ is in agreement with Neumann's result. Neumann's temperature distribution in the solid, namely

$$T(z, t) = \frac{T_m}{\operatorname{erf}\lambda}\operatorname{erf}\frac{z}{2\sqrt{(\kappa t)}} \tag{51}$$

is obtained immediately by substituting $\bar{B}_0$ into equation (45).

4.2. *Example for Section 4.1*

Equations (47) will now be solved for the particular case in which $h = 0$ and the heat flux $Q(x, t)$ is given by

$$Q(x, t) = Q_0\left(1 + C\cos\frac{\pi x}{a}\right) \qquad t > 0, \tag{52a}$$

where Q_0 and C are constants satisfying

$$|C| \leq 1 \tag{52b}$$

insuring that the problem is one of class (1).

With (52a) and the appropriate Green's function, i.e. equation (31), equations (47a, b) become

$$\begin{aligned} &\sum_{i=0}^{\infty}\frac{\exp[-\kappa(i\pi/a)^2 t]}{2\sqrt{(\pi\kappa t)}}\int_0^{\infty}\bar{B}_i(z')\left\{\exp\left[-\frac{(\mathscr{S}-z')^2}{4\kappa t}\right]+\exp\left[-\frac{(\mathscr{S}+z')^2}{4\kappa t}\right]\right\}dz'\cos\frac{i\pi x}{a} \\ &\quad -\frac{2Q_0\sqrt{(\kappa t)}}{k}\operatorname{ierfc}\frac{\mathscr{S}}{2\sqrt{(\kappa t)}}-\frac{Q_0 Ca}{2k\pi}\left[\exp(-\pi\mathscr{S}/a)\left(1+\operatorname{erf}\left[\frac{\pi}{a}\sqrt{(\kappa t)}-\frac{\mathscr{S}}{2\sqrt{(\kappa t)}}\right]\right)\right. \\ &\quad \left.-\exp(\pi\mathscr{S}/a)\left(1-\operatorname{erf}\left[\frac{\pi}{a}\sqrt{(\kappa t)}+\frac{\mathscr{S}}{2\sqrt{(\kappa t)}}\right]\right)\right]\cos\frac{\pi x}{a} = T_m, \end{aligned} \tag{53a}$$

$$\begin{aligned} &\left[1+\left(\frac{\partial\mathscr{S}}{\partial x}\right)^2\right]\left\{-k\sum_{i=0}^{\infty}\frac{\exp[-\kappa(i\pi/a)^2 t]}{4\sqrt{\pi}(\kappa t)^{\frac{3}{2}}}\int_0^{\infty}\bar{B}_i(z')\left\{(\mathscr{S}-z')\exp\left[-\frac{(\mathscr{S}-z')^2}{4kt}\right]\right.\right. \\ &\quad \left.+(\mathscr{S}+z')\exp\left[-\frac{(\mathscr{S}+z')^2}{4\kappa t}\right]\right\}dz'\cos\frac{i\pi x}{a}+Q_0\operatorname{erfc}\frac{\mathscr{S}}{2\sqrt{(\kappa t)}} \\ &\quad +\frac{Q_0 C}{2}\left[\exp(-\pi\mathscr{S}/a)\left(1+\operatorname{erf}\left[\frac{\pi}{a}\sqrt{(\kappa t)}-\frac{\mathscr{S}}{2\sqrt{(\kappa t)}}\right]\right)+\exp(\pi\mathscr{S}/a)\left(1-\operatorname{erf}\left[\frac{\pi}{a}\sqrt{(\kappa t)}+\frac{\mathscr{S}}{2\sqrt{(\kappa t)}}\right]\right)\right] \\ &\quad \left.\cos\frac{\pi x}{a}\right\} = -\rho_S L\frac{\partial\mathscr{S}}{\partial t}. \end{aligned} \tag{53b}$$

These equations can be put in dimensionless form by means of the following non-dimensional variables and parameters:

$$y = \frac{t}{a^2/4\kappa}; \quad \xi = \frac{\sigma}{a}; \quad E_i = \frac{\bar{E}_i}{a}; \quad B_i(\zeta) = \frac{\bar{B}_i}{T_m};$$
$$\rho = \frac{x}{a}; \quad \zeta = \frac{z}{a}; \quad p = \frac{Q_0 a}{kT_m}; \quad m = \frac{Q_0 a}{2\rho_S \kappa L} \tag{54}$$

and then become:

$$\sum_{i=0}^{\infty} \frac{\exp[-(i\pi/2)^2 y]}{\sqrt{(\pi y)}} \int_0^{\infty} B_i(\zeta')\left\{\exp\left[-\frac{(\xi-\zeta')^2}{y}\right]+\exp\left[-\frac{(\xi+\zeta')^2}{y}\right]\right\} d\zeta' \cos i\pi\rho$$
$$= 1+p\sqrt{y}\,\text{ierfc}\frac{\xi}{\sqrt{y}}+\frac{Cp}{2\pi}\left[\exp(-\pi\xi)\left(1+\text{erf}\left[\frac{\pi}{2}\sqrt{y}-\frac{\xi}{\sqrt{y}}\right]\right)\right.$$
$$\left.-\exp(\pi\xi)\left(1-\text{erf}\left[\frac{\pi}{2}\sqrt{y}+\frac{\xi}{\sqrt{y}}\right]\right)\right]\cos\pi\rho, \tag{55a}$$

$$\left[1+\left(\frac{\partial\xi}{\partial\rho}\right)^2\right]\frac{2}{\sqrt{\pi p y^{\frac{3}{2}}}}\sum_{i=0}^{\infty}\exp\left[-\left(\frac{i\pi}{2}\right)^2 y\right]\int_0^{\infty} B_i(\zeta')\left\{(\xi-\zeta')\exp\left[-\frac{(\xi-\zeta')^2}{y}\right]\right.$$
$$\left.+(\xi+\zeta')\exp\left[-\frac{(\xi+\zeta')^2}{y}\right]\right\}d\zeta'\cos i\pi\rho = \frac{2}{m}\frac{\partial\xi}{\partial y}$$
$$+\left[1+\left(\frac{\partial\xi}{\partial\rho}\right)^2\right]\left[\text{erfc}\frac{\xi}{\sqrt{y}}+\frac{C}{2}\left\{\exp(-\pi\xi)\left(1+\text{erf}\left[\frac{\pi}{2}\sqrt{y}-\frac{\xi}{\sqrt{y}}\right]\right)\right.\right.$$
$$\left.\left.+\exp(\pi\xi)\left(1-\text{erf}\left[\frac{\pi}{2}\sqrt{y}+\frac{\xi}{\sqrt{y}}\right]\right)\right\}\cos\pi\rho\right], \tag{55b}$$

while the conditions on the temperature and the melt thickness (46) and (47c) respectively become:

$$B_0(0) = 1,$$
$$B_i(0) = 0 \qquad i = 1, 2, 3, \dots, \tag{55c}$$
$$E_i(0) = 0 \qquad i = 0, 1, 2, \dots. \tag{55d}$$

As was the case for the melting problem of Section 3, a short time solution will be found. Proceeding as before, the first term of the short time solution is obtained, a general series is then assumed and the coefficients of like cosines and like powers of y are matched to obtain the unknown constants in the series.

The right of (55a) expanded in y is:† $1+p/\sqrt{\pi}\sqrt{y}+(Cp)/\sqrt{\pi}\sqrt{y}\cos\pi\rho+\dots$. If only the first term is retained, and since $\exp[-(i\pi/2)^2 y] \cong 1$ for small y, this equation reduces

† This expansion is valid since, as follows from equations (7b),

$$\lim_{y\to 0} \xi(y)/\sqrt{y} \to 0.$$

The alternative procedure of [21] can also be used to give the same result.

to

$$\sum_{i=0}^{\infty} \frac{1}{\sqrt{(\pi y)}} \int_0^{\infty} B_i(\zeta')\left\{\exp\left[-\frac{(\xi-\zeta')^2}{y}\right]+\exp\left[-\frac{(\xi+\zeta')^2}{y}\right]\right\} \mathrm{d}\zeta' \cos i\pi\rho = 1. \tag{56a}$$

The solution to this set of equations satisfying (55c) is readily verified to be

$$\begin{aligned} B_0(\zeta) &= 1, \\ B_i(\zeta) &= 0 \qquad i = 1, 2, 3, \ldots . \end{aligned} \tag{56b}$$

With these values of $B_i(\zeta)$'s the left-hand side of (55b) becomes zero. The right-hand side expanded in y is

$$\frac{2}{m}\frac{\partial \xi}{\partial y}+1-\frac{2}{\sqrt{\pi}}\frac{\xi}{\sqrt{y}}+\ldots+\left[C-\frac{2C}{\sqrt{\pi}}\frac{\xi}{\sqrt{y}}+\ldots\right]\cos \pi\rho.$$

Retaining terms of the lowest order only, one obtains the following set of differential equations:

$$\frac{2}{m}\sum_{i=0}^{\infty}\frac{\partial E_i(y)}{\partial y}\cos i\pi\rho = -(1+C\cos\pi\rho) \tag{57a}$$

whose solution under initial condition (55d) is

$$\begin{aligned} E_0(y) &= \frac{m}{2}y, \\ E_1(y) &= \frac{mC}{2}y, \\ E_i(y) &= 0 \qquad i = 2, 3, 4, \ldots , \end{aligned} \tag{57b}$$

so that, for very short times

$$\xi(\rho, y) = \frac{m}{2}\, y(1+C\cos\pi\rho)+\ldots y \ll 1, \tag{57c}$$

in agreement with the result given in equation (7b).

To extend the validity of the solution in the time domain, consider the following series:

$$\begin{aligned} \frac{T^*(\rho,\zeta)}{T_m} &= 1+\sum_{i=0}^{\infty}(a_{i1}\zeta+a_{i2}\zeta^2+a_{i3}\zeta^3+\ldots)\cos i\pi\rho, \\ \xi(\rho, y) &= \frac{m}{2}y(1+C\cos\pi\rho)+\sum_{i=0}^{\infty}(b_{i1}y^{\frac{3}{2}}+b_{i2}y^2+b_{i3}y^{\frac{5}{2}}+\ldots)\cos i\pi\rho. \end{aligned} \tag{58}$$

A formal solution is constructed by substituting these series into equations (55a, b) and by equating coefficients of like cosines and like powers of y. The unknown initial temperature distribution and thickness of the solid are finally given by:

$$\begin{aligned} \frac{T^*}{T_m}(\rho,\zeta) = {} & \left[1+p\zeta-mp\left(1+\frac{C^2}{2}\right)\zeta^2+\ldots\right]+\left(Cp\zeta-2mpC\zeta^2+\frac{\pi^2Cp}{6}\zeta^3+\ldots\right)\cos\pi\rho \\ & +\left(-\frac{mpC^2}{2}\zeta^2+\ldots\right)\cos 2\pi\rho+\ldots , \end{aligned} \tag{59a}$$

$$\xi(\rho, y) = \left[\frac{m}{2}y - \frac{m^3}{4}\left(1+\frac{3C^2}{2}\right)y^2 + \dots\right] + \left[\frac{mC}{2}y - \frac{m^3}{4}\left(3C+\frac{3C^3}{4}\right)y^2 + \dots\right]\cos\pi\rho$$
$$+\left(-\frac{3}{8}m^3C^2y^2+\dots\right)\cos 2\pi\rho + \left(-\frac{m^3C^3}{16}y^2+\dots\right)\cos 3\pi\rho + \dots \tag{59b}$$

The temperature field obtained by substituting the fictitious initial temperature distribution into (45), or, more conveniently, from (55a) is

$$\frac{T_E}{T_m}(\rho,\zeta,y) = \left[1+p\zeta - mp\left(1+\frac{C^2}{2}\right)\left(\frac{y}{2}+\zeta^2\right)+\dots\right] + \left[-\frac{Cp}{2\pi}\left\{e^{-\pi\zeta}\left(1+\text{erf}\left[\frac{\pi}{2}\sqrt{y}-\frac{\zeta}{\sqrt{y}}\right]\right)\right.\right.$$
$$\left.-e^{\pi\zeta}\left(1-\text{erf}\left[\frac{\pi}{2}\sqrt{y}+\frac{\zeta}{\sqrt{y}}\right]\right)\right\} + Cp\sqrt{\frac{y}{\pi}}\exp\left[-\left(\frac{\pi}{2}\right)^2 y + \frac{\zeta^2}{y}\right] + Cp\zeta\exp\left[-\left(\frac{\pi}{2}\right)^2 y\right]\text{erf}\frac{\zeta}{\sqrt{y}}$$
$$-2mpC\exp\left[-\left(\frac{\pi}{2}\right)^2 y\right]\left(\frac{y}{2}+\zeta^2\right) + \frac{\pi^2 Cp}{6}\exp\left[-\left(\frac{\pi}{2}\right)^2 y\right]\left\{\sqrt{\frac{y}{\pi}}(y+\zeta^2)\exp\left[-\left(\frac{\zeta^2}{y}\right)\right]\right.$$
$$\left.\left.+\zeta\left(\frac{3}{2}y+\zeta^2\right)\text{erf}\frac{\zeta}{\sqrt{y}}\right\}+\dots\right]\cos\pi\rho + \left[-\frac{mpC^2}{2}e^{-\pi^2 y}\left\{\frac{y}{2}+\zeta^2\right\}+\dots\right]\cos 2\pi\rho + \dots, \tag{60}$$

which, of course, has physical meaning only for $0 < z < s(x, t)$.

The solution of the problem is now complete. Curves showing the variation of thickness of the solid with x and t are presented in Fig. 5. The temperature along the axis of symmetry is plotted for several values of time in Fig. 6, and the temperature field at a fixed time showing the variation in x is plotted in Fig. 7. The dotted portions of the curves of Figs. 6 and 7 represent the temperature field in the fictitious region, $z > s(x, t)$.

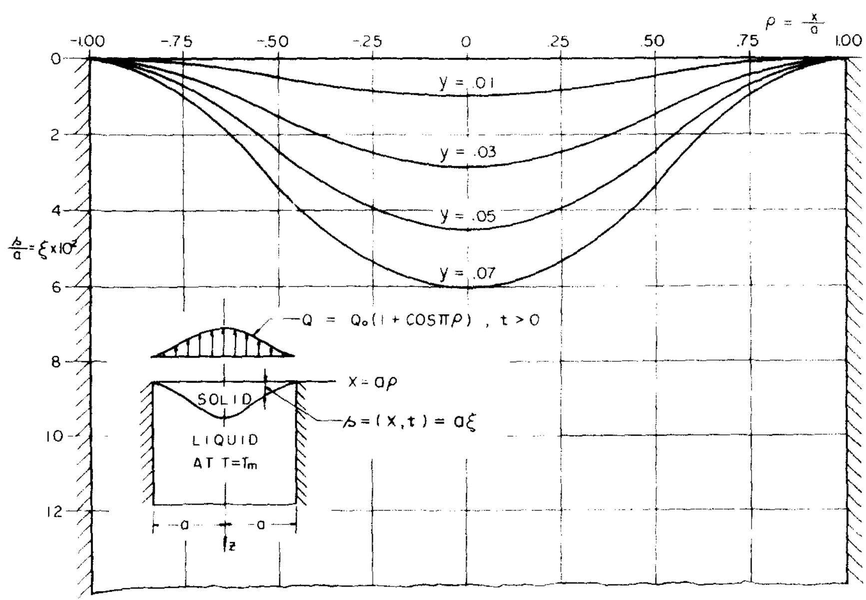

FIG. 5. A dimensionless plot of the solid thickness vs. x/a for various values of time and for $m = p = C = 1$.

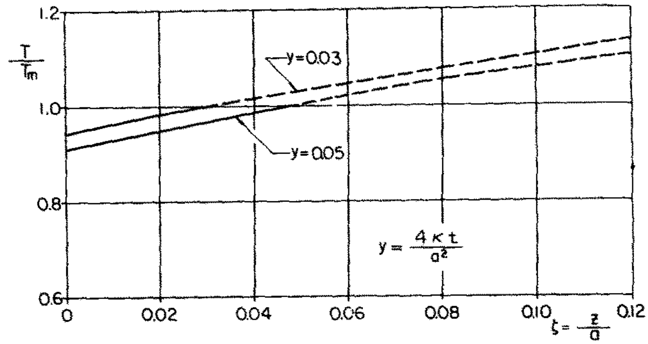

FIG. 6. A dimensionless plot of the temperature field for the solidification problem vs. z/a along the axis of symmetry for various times and for $m = p = C = 1$.

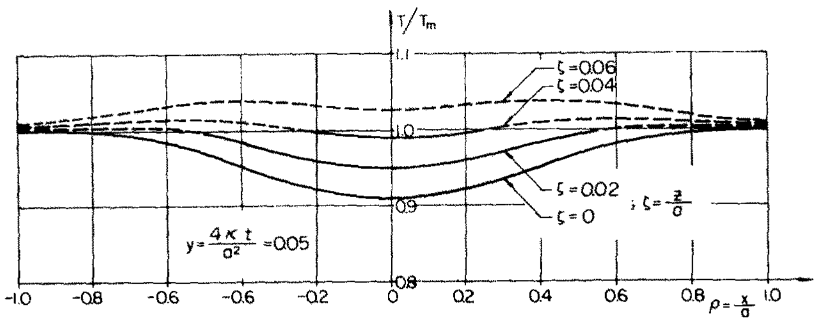

FIG. 7. A dimensionless plot of the temperature field for the solidification problem vs. x/a for $y = 0{\cdot}05$, at various depths in the strip and for $m = p = C = 1$.

4.3. *Discussion of the solution of Section 4.2*

The following remarks concerning the solution presented in the preceding section are of interest.

(1) The one-dimensional special case in which $C = 0$ has been solved by Evans *et al.* [20]. The thickness of the solid in this case is, from equation (59b),

$$\xi(y) = \frac{m}{2}y - \frac{m^3}{4}y^2 + \ldots \tag{61}$$

in agreement with [20].

(2) The remarks, given in Section 3.3(2), concerning the character of the terms of the Fourier series and of the powers of y appearing in the Fourier coefficients are again valid here. Note that in the present solution only integral powers of y appear in the series given.

(3) As in the example of Section 3, the contribution of the last term of the series considered in the numerical calculations was small compared to that of the other terms in the series.

(4) Note that the thickness of the solid is extremely small at $x = \pm a$ in Fig. 5. The slow growth of the solid at these points is due to the vanishing of thermal gradients there.

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Zusammenfassung—Zweidimensionale Schmelz- und Erstarrungsprobleme werden in drei zweckdienliche Klassen engeteilt, und eine allgemeine Methode zur Lösung der in eine dieser Klassen fallenden Probleme wird angegeben. Bei dieser Methode verwendet man das Konzept eines fiktiven Körpers konstanter Geometrie, in welchen der wirkliche Körper eingebettet ist (die Abmessungen des letzteren ändern sich mit der Phasenlage). Der fiktive Körper befindet sich unter der Einwirkung eines fiktiven Wärmeflusses oder (bei einigen Problemen) einer fiktiven anfänglichen Temperaturverteilung. Das in diesen unbekannten, fiktiven Grössen formulierte Problem führt zu einem Integrodifferentialgleichungssystem mit mehreren Unbekannten, welches, entweder zahlenmässig, oder in Reihenform gelöst werden muss. Zwei Probleme werden im Einzelnen behandelt. In dem ersten Problem wird das Schmelzen eines begrenzten, isolierten Blockes formuliert, von dem die Schmelze-sofort entfernt wird, und ein Beispiel für einen einseitig begrenzten, isolierten Streifen wird angegeben. In dem zweiten Problem wird das Erstarren eines begrenzten, isolierten Blockes formuliert, der nicht überhitzt wurde, und ein auf einen bestimmten früheren Abkühlvorgang zutreffendes Beispiel wird angegeben. Bei beiden Problemen werden zweidimensionale Effekte durch räumliche Abhängigkeit der Erwärmungs- und Abkühlungsbedingungen hervorgerufen, und die Lösungen in kurzzeitiger Reihenform entwickelt.

Абстракт—Проблемы в двух измерениях относительно плавления и затвердевания удобно делятся на три класса, итакже дается общий метод для решения проблем одного из этих классов. Этот метод утилизирует понятие фиктивного тела с постоянной геометрией, в которое включено реальное тело (измерения которого меняются в зависимости от изменения фазы); на фиктивное тело действует фиктивный поток теплоты или, в некоторых проблемах, фиктивное начальное распределение температуры. Таким образом проблема, формулированная в этих неизвестных фиктивных величин, дает в результате группу интегро-дифференциальных уравнений для совместного решения, численно или в форме рядов. Две проблемы рассмотрены подробно. В первой проблеме выражается в формуле плавление конечной изолированной плиты при условии немедленного удаления расплавленной фазы, дается пример для полубесконечной изолированной планки. Во второй проблеме выражается формулой затвердение конечной изолированной плиты с нулевым перегревом и дается пример специфической истории охлаждения. В обеих проблемах представлены дву-мерные эффекты посредством пространственных вариаций условий нагревания или охлаждения и выведены решения коротко-временными рядами.